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A Hosszú-Gluskin Algebra and a Central Operation of (sm, m)-Groups

Anita Katić[a], Mališa R. Žižović[b]

[a]*Faculty of Electrical Engineering, University of Osijek, Kneza Trpimira 2b, 31000 Osijek, Croatia*
[b]*University Singidunum, Danijelova 32, 11000 Belgrade, Serbia*

Abstract. In this paper we prove a generalization of the Hosszú-Gluskin theorem for (sm, m)-groups in terms of a $\{1, (s-1)m+1\}$-neutral operation and we define the algebra $\left(Q^m, \left\{\cdot, \varphi, c_1^m\right\}\right)$ associated to the (sm, m)-group (Q, A). The central operation of an (n, m)-group is defined in [7]. Research results of central operation properties using a bijection $\sigma_\alpha : Q^m \to Q^m$ are presented by means of a series of theorems. Then, a central operation of an (sm, m)-group is investigated using the previously mentioned algebra.

1. Introduction

Firstly, we explain a notation introduced by Janez Ušan which we use in this paper. Let $p \in N$ and let $q \in N_0$. Then a_p^q denotes $a_p, \ldots, a_q$ the sequence of elements of a set Q if $p < q$. If $p = q$, then a_p^q denotes the element a_p into the set Q and if $p > q$, then a_p^q denotes the empty sequence. If a_p^q is a sequence over a set Q, $p \leq q$ and the equalities $a_p = \ldots = a_q = a$ are satisfied, then a_p^q is denoted by $\overset{q-p+1}{a}$. If we have $a_1^m \in Q^m$, then $\overline{a_1^m\Big|}^{\,s}$ denotes a sequence of s sequences a_1^m.

The notion of an (n, m)-group, as a generalization of the notion of an n-group, that is of a group was introduced by Ǵ. Čupona in 1983. Let Q be a nonempty set and let A be a mapping of the set Q^n into the set Q^m, where $n \geq m+1$. Then, we say that (Q, A) is an (n, m)-groupoid. Since every groupoid is a group if and only if it is a semigroup and a quasigroup, similarly an (n, m)-group was defined as an (n, m)-semigroup and an (n, m)-quasigroup.

Definition 1.1. *[1] Let (Q, A) be an (n, m)-groupoid, $n \geq m+1$. (Q, A) is an (n, m)-group iff the following statements hold:*

a) $\forall i, j \in \{1, \ldots, n-m+1\}$ and for every sequence $x_1^{2n-m} \in Q$ the following equality holds:

$$A\left(x_1^{i-1}, A\left(x_i^{i+n-1}\right), x_{i+n}^{2n-m}\right) = A\left(x_1^{j-1}, A\left(x_j^{j+n-1}\right), x_{j+n}^{2n-m}\right),$$

which is called an $\langle i, j\rangle$-associative law.

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Email addresses: anita.katic@etfos.hr (Anita Katić), zizovic@gmail.com (Mališa R. Žižović)

b) $\forall i \in \{1, \dots, n-m+1\}$ *and for every sequence* $a_1^n \in Q$ *there is exactly one sequence* $x_1^m \in Q$ *such that the following equality holds:*

$$A\left(a_1^{i-1}, x_1^m, a_i^{n-m}\right) = a_{n-m+1}^n.$$

An (n,m)-groupoid (Q,A), where the statement *a)* holds, is called an (n,m)-semigroup, so that the (n,m)-groupoid (Q,A), where the statement *b)* holds, is called a weak (n,m)-quasigroup.

Example 1.2. *Let* $(Q,\cdot)$, $Q = \{1,2,3,4\}$ *be the Klein group defined with the following table:*

·	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Let ψ *be the permutation of the set Q defined in the following way:* $\psi \overset{def}{=} \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,4\,3 \end{pmatrix}$. *Further on, let* $A : Q^6 \to Q^2$ *be the mapping defined in the following way:* $A(x_1^6) \overset{def}{=} (x_1 \cdot \psi(x_3) \cdot x_5, x_2 \cdot \psi(x_4) \cdot x_6)$. *We prove that* (Q,A) *is a* $(6,2)$*-group. Firstly, we prove that the* $\langle 1,2\rangle$*-associative law holds.*

$$A(A(x_1^6), x_7^{10}) = A(x_1 \cdot \psi(x_3) \cdot x_5, x_2 \cdot \psi(x_4) \cdot x_6, x_7, x_8, x_9, x_{10}) = (x_1 \cdot \psi(x_3) \cdot x_5 \cdot \psi(x_7) \cdot x_9, x_2 \cdot \psi(x_4) \cdot x_6 \cdot \psi(x_8) \cdot x_{10})$$
$$A(x_1, A(x_2^7), x_8^{10}) = A(x_1, x_2 \cdot \psi(x_4) \cdot x_6, x_3 \cdot \psi(x_5) \cdot x_7, x_8, x_9, x_{10}) = (x_1 \cdot \psi(x_3) \cdot x_5 \cdot \psi(x_7) \cdot x_9, x_2 \cdot \psi(x_4) \cdot x_6 \cdot \psi(x_8) \cdot x_{10})$$

Similarly, we can prove that the $\langle i,j\rangle$*-associative law holds for all* $i, j \in \{1,2,3,4,5\}$.

Now, we prove that the statement b) of Definition 1.1 for $i = 1$ *holds.*

$$A(x_1^2, a_1^4) = a_5^6 \Leftrightarrow (x_1 \cdot \psi(a_1) \cdot a_3, x_2 \cdot \psi(a_2) \cdot a_4) = (a_5, a_6) \Leftrightarrow x_1 \cdot \psi(a_1) \cdot a_3 = a_5 \text{ and } x_2 \cdot \psi(a_2) \cdot a_4 = a_6$$

Because $(Q,\cdot)$ *is a group, for every sequence* $a_1^6 \in Q$ *there is exactly one* $x_1 \in Q$ *and exactly one* $x_2 \in Q$ *such that the previous sequence of equivalences holds. Likewise, we can prove that the statement b) of Definition 1.1 for all* $i \in \{1,2,3,4,5\}$ *holds.*

An interesting research method of n-structures and (n,m)-structures was inspired by J. Ušan, who in his manifold papers puts emphasis on a neutral operation. These results are systematized in the monograph [16]: n-groups in the light of the neutral operations. An $\{i,j\}$-neutral operation of an n-groupoid (Q,A) was defined by Ušan in 1988 [10] and it presents a generalization of a neutral element of a groupoid. Then, in 1989 he defined a $\{1, n-m+1\}$-neutral operation of an (n,m)-groupoid (Q,A).

Definition 1.3. *[11] Let* $n \geq 2m$ *and let* (Q,A) *be an* (n,m)*-groupoid. Furthermore, let* e_L, e_R *and e be mappings of the set* Q^{n-2m} *into the set* Q^m*. Then:*

a) e_L *is a left* $\{1, n-m+1\}$*-neutral operation of the* (n,m)*-groupoid* (Q,A) *iff* $\forall a_1^{n-2m}, x_1^m \in Q$ *the following equality holds:*

$$A\left(e_L\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = x_1^m;$$

b) e_R *is a right* $\{1, n-m+1\}$*-neutral operation of the* (n,m)*-groupoid* (Q,A) *iff* $\forall a_1^{n-2m}, x_1^m \in Q$ *the following equality holds:*

$$A\left(x_1^m, a_1^{n-2m}, e_R\left(a_1^{n-2m}\right)\right) = x_1^m;$$

c) e is a $\{1, n-m+1\}$*-neutral operation of the* (n,m)*-groupoid* (Q,A) *iff it is a left* $\{1, n-m+1\}$*-neutral operation and a right* $\{1, n-m+1\}$*-neutral operation.*

Example 1.4. *Let (Q, A) be the $(6, 2)$-group defined in Example 1.2. Now we define a mapping $e : Q^2 \to Q^2$ such that the following equality holds: $e(a_1^2) \overset{def}{=} (\psi(a_1), \psi(a_2))$. We prove that e is a $\{1, 5\}$-neutral operation of the $(6, 2)$-group (Q, A). $\forall a_1^2, x_1^2 \in Q$ two following sequences of equalities hold:*

$$A(e(a_1^2), a_1^2, x_1^2) = A(\psi(a_1), \psi(a_2), a_1^2, x_1^2) = (\psi(a_1) \cdot \psi(a_1) \cdot x_1, \psi(a_2) \cdot \psi(a_2) \cdot x_2) = (x_1, x_2),$$
$$A(x_1^2, a_1^2, e(a_1^2)) = A(x_1^2, a_1^2, \psi(a_1), \psi(a_2)) = (x_1 \cdot \psi(a_1) \cdot \psi(a_1), x_2 \cdot \psi(a_2) \cdot \psi(a_2)) = (x_1, x_2).$$

If we put $(n, m) = (2, 1)$, the definition of a $\{1, n - m + 1\}$-neutral operation of an (n, m)-groupoid is the same as the definition of a neutral element of a groupoid. Furthermore, a $\{1, n-m+1\}$-neutral operation has the same properties as a neutral element in binary structures. Some of them are the following statements:

- there is at most one $\{1, n - m + 1\}$-neutral operation of the (n, m)-groupoid (Q, A) $(n \geq 2m)$;
- if e_L is a left $\{1, n - m + 1\}$-neutral operation of the (n, m)-groupoid (Q, A) and e_R is a right $\{1, n - m + 1\}$-neutral operation of the (n, m)-groupoid (Q, A), then they are equal and $e = e_L = e_R$ is a $\{1, n - m + 1\}$-neutral operation of the (n, m)-groupoid (Q, A);
- every (n, m)-group, where $n \geq 2m$, has exactly one $\{1, n - m + 1\}$-neutral operation.

The previous statements were proved in [11].

Moreover, one generalization of an inverse element in binary structures was defined in n-structures by Janez Ušan 1994 in [12], in terms of a neutral operation. Similarly, he defined an inverse operation in (n, m)-structures.

Proposition 1.5. *[14] Let (Q, A) be an (n, m)-groupoid, $n \geq 2m$ and let the following statements hold:*

(a) in (Q, A) a $\langle 1, n - m + 1\rangle$-associative law holds;
(b) for every sequence $a_1^n \in Q$ there is at least one $x_1^m \in Q^m$ such that the following equality holds:

$$A\left(a_1^{n-m}, x_1^m\right) = a_{n-m+1}^n;$$

(c) for every sequence $a_1^n \in Q$ there is at least one $y_1^m \in Q^m$ such that the following equality holds:

$$A\left(y_1^m, a_1^{n-m}\right) = a_{n-m+1}^n.$$

Then, there are mappings $^{-1} : Q^{n-m} \to Q^m$ and $e : Q^{n-2m} \to Q^m$ such that in the algebra $(Q, \{A, ^{-1}, e\})$ the following equalities hold:

(i) $A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, A\left(b_1^m, a_1^{n-2m}, x_1^m\right)\right) = x_1^m$;

(ii) $A\left(A\left(x_1^m, a_1^{n-2m}, b_1^m\right), a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = x_1^m$;

(iii) $A\left(b_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, b_1^m\right)^{-1}\right) = e\left(a_1^{n-2m}\right)$;

(iv) $A\left(\left(a_1^{n-2m}, b_1^m\right)^{-1}, a_1^{n-2m}, b_1^m\right) = e\left(a_1^{n-2m}\right)$.

Where holds

$$\left(a_1^{n-2m}, b_1^m\right)^{-1} \overset{def}{=} E\left(a_1^{n-2m}, b_1^m, a_1^{n-2m}\right), \forall a_1^{n-2m}, b_1^m \in Q,$$

where E is a $\{1, 2n-2m+1\}$-neutral operation of the $(2n-m, m)$-groupoid $(Q, \overset{2}{A})$ and $\overset{2}{A}\left(x_1^{2n-m}\right) \overset{def}{=} A\left(A\left(x_1^n\right), x_{n+1}^{2n-m}\right)$.

In two following propositions, the equalities, which hold for a $\{1, n - m + 1\}$-neutral operation $(n > 2m)$, are given and they are important for further research of (n, m)-groups.

Proposition 1.6. *[5] Let (Q, A) be an (n, m)-group, $n > 2m$ and let e be its $\{1, n-m+1\}$-neutral operation. Then $\forall a_1^{n-2m} \in Q$, $\forall x_1^m \in Q^m$ and $\forall t \in \{1, \dots n-2m+1\}$ the following equalities hold:*

$$A\left(x_1^m, a_t^{n-2m}, e\left(a_1^{n-2m}\right), a_1^{t-1}\right) = x_1^m;$$

$$A\left(a_t^{n-2m}, e\left(a_1^{n-2m}\right), a_1^{t-1}, x_1^m\right) = x_1^m.$$

Proposition 1.7. *[6] Let (Q, A) be an (n, m)-group, $n \geq 2m$, e its $\{1, n-m+1\}$-neutral operation and let $^{-1} : Q^{n-m} \to Q^m$ be its inverse operation. Then $\forall a_1^{n-2m}, b_1^{n-2m}, x_1^m, y_1^m \in Q$ the following equality holds:*

$$A\left(x_1^m, b_1^{n-2m}, y_1^m\right) = A\left(A\left(x_1^m, a_1^{n-2m}, \left(a_1^{n-2m}, e\left(b_1^{n-2m}\right)\right)^{-1}\right), a_1^{n-2m}, y_1^m\right).$$

2. A Generalization of the Hosszú-Gluskin Theorem for Some (n, m)-Groups in Terms of a Neutral Operation

Hosszú-Gluskin theorem is very important for the description and systematization of n-groups.

Theorem 2.1. *[8], [9] For every n-group (Q, A), $n \geq 3$, there is an algebra $(Q, \{\cdot, \varphi, b\})$ such that the following statements hold:* (1) *$(Q, \cdot)$ is a group;* (2) *$\varphi \in Aut(Q, \cdot)$;* (3) *$\varphi(b) = b$;* (4) *for every $x \in Q$, $\varphi^{n-1}(x) \cdot b = b \cdot x$;* (5) *for every $x_1^n \in Q, A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$.*

In 1995 Ušan proved the Hosszú-Gluskin theorem by using a neutral operation and defined a Hosszú-Gluskin algebra of order n, $n \geq 3$, [13]. One generalization of the Hosszú-Gluskin theorem for (sm, m)-groups was described by Čupona and others in 1988 [2].

Following Ušan's approach to the research of (n, m)-structures, a generalization of the Hosszú-Gluskin theorem for (sm, m)-groups, $s > 2$ in terms of a $\{1, n-m+1\}$-neutral operation was proved in the following two theorems.

Theorem 2.2. *Let (Q, A) be an (sm, m)-group, $s > 2$, $e : Q^{(s-2)m} \to Q^m$ its $\{1, (s-1)m+1\}$-neutral operation and let $a_1^{(s-2)m} \in Q$ be an arbitrary sequence. $\forall x_1^m, y_1^m \in Q^m$, we define the following operations:*

(a) $x_1^m \cdot y_1^m \overset{def}{=} A\left(x_1^m, a_1^{(s-2)m}, y_1^m\right)$,

(b) $\varphi\left(x_1^m\right) \overset{def}{=} A\left(e\left(a_1^{(s-2)m}\right), x_1^m, a_1^{(s-2)m}\right)$,

(c) $c_1^m \overset{def}{=} A\left(\overline{\overset{s}{e\left(a_1^{(s-2)m}\right)}}\right)$.

Then, the following statements hold:

(i) $(Q^m, \cdot)$ *is a group;*

(ii) $\varphi \in Aut\left(Q^m, \cdot\right)$;

(iii) $\varphi\left(c_1^m\right) = c_1^m$;

(iv) $\varphi^{s-1}\left(b_1^m\right) \cdot c_1^m = c_1^m \cdot b_1^m$, $\forall b_1^m \in Q^m$;

(v) $A\left(x_1^{sm}\right) = x_1^m \cdot \varphi\left(x_{m+1}^{2m}\right) \cdot \varphi^2\left(x_{2m+1}^{3m}\right) \cdots \varphi^{s-1}\left(x_{(s-1)m+1}^{sm}\right) \cdot c_1^m$, $\forall x_1^{sm} \in Q$.

Proof. (i) $\forall x_1^m, y_1^m, z_1^m \in Q^m$, according to (a) and according to definition of the (sm, m)-group, the following sequence of equalities holds:

$$x_1^m \cdot \left(y_1^m \cdot z_1^m\right) = A\left(x_1^m, a_1^{(s-2)m}, A\left(y_1^m, a_1^{(s-2)m}, z_1^m\right)\right) = A\left(A\left(x_1^m, a_1^{(s-2)m}, y_1^m\right), a_1^{(s-2)m}, z_1^m\right) = \left(x_1^m \cdot y_1^m\right) \cdot z_1^m,$$

which proves that $(Q^m, \cdot)$ is a semigroup.

$\forall x_1^m, y_1^m \in Q^m$ and for an arbitrary sequence $a_1^{(s-2)m} \in Q$, since (Q, A) is an (sm, m)-group, there is exactly one $z_1^m \in Q^m$ and there is exactly one $k_1^m \in Q^m$ such that the following implication holds:

$$\left.\begin{array}{l} A\left(x_1^m, a_1^{(s-2)m}, z_1^m\right) = y_1^m \overset{(a)}{\Leftrightarrow} x_1^m \cdot z_1^m = y_1^m \\ A\left(k_1^m, a_1^{(s-2)m}, x_1^m\right) = y_1^m \overset{(a)}{\Leftrightarrow} k_1^m \cdot x_1^m = y_1^m \end{array}\right\} \Rightarrow (Q^m, \cdot) \text{ is a quasigroup.}$$

(ii) By (b) and since (Q, A) is an (sm, m)-quasigroup, we conclude that $\varphi : Q^m \to Q^m$ is a bijection. We prove that φ is a homomorphism.

$$\begin{aligned} \varphi\left(x_1^m \cdot y_1^m\right) &\overset{(b),(a)}{=} A\left(e\left(a_1^{(s-2)m}\right), A\left(x_1^m, a_1^{(s-2)m}, y_1^m\right), a_1^{(s-2)m}\right) \overset{1.1}{=} A\left(A\left(e\left(a_1^{(s-2)m}\right), x_1^m, a_1^{(s-2)m}\right), y_1^m, a_1^{(s-2)m}\right) \overset{1.6}{=} \\ &= A\left(A\left(e\left(a_1^{(s-2)m}\right), x_1^m, a_1^{(s-2)m}\right), A\left(a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right), y_1^m\right), a_1^{(s-2)m}\right) \overset{1.1}{=} \\ &= A\left(A\left(e\left(a_1^{(s-2)m}\right), x_1^m, a_1^{(s-2)m}\right), a_1^{(s-2)m}, A\left(e\left(a_1^{(s-2)m}\right), y_1^m, a_1^{(s-2)m}\right)\right) \overset{(a),(b)}{=} \varphi\left(x_1^m\right) \cdot \varphi\left(y_1^m\right). \end{aligned}$$

(iii)

$$\begin{aligned} \varphi\left(c_1^m\right) &\overset{(b)}{=} A\left(e\left(a_1^{(s-2)m}\right), c_1^m, a_1^{(s-2)m}\right) \overset{(c)}{=} A\left(e\left(a_1^{(s-2)m}\right), A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right), a_1^{(s-2)m}\right) \overset{1.1}{=} \\ &= A\left(A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right), e\left(a_1^{(s-2)m}\right), a_1^{(s-2)m}\right) \overset{1.6}{=} A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{(c)}{=} c_1^m. \end{aligned}$$

(iv) $\forall b_1^m \in Q^m$ the following sequence of equalities holds:

$$\begin{aligned} c_1^m \cdot b_1^m &\overset{(a),(c)}{=} A\left(A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right), a_1^{(s-2)m}, b_1^m\right) \overset{1.1}{=} A\left(\overset{s-1}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(e\left(a_1^{(s-2)m}\right), a_1^{(s-2)m}, b_1^m\right)\right) \overset{1.3}{=} \\ &= A\left(\overset{s-1}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(b_1^m, a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right)\right) \overset{1.1}{=} A\left(\overset{s-2}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(e\left(a_1^{(s-2)m}\right), b_1^m, a_1^{(s-2)m}\right), e\left(a_1^{(s-2)m}\right)\right) \overset{(b)}{=} \\ &= A\left(\overset{s-2}{\overline{e\left(a_1^{(s-2)m}\right)}}, \varphi\left(b_1^m\right), e\left(a_1^{(s-2)m}\right)\right) \overset{1.3}{=} A\left(\overset{s-2}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(\varphi\left(b_1^m\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{1.1}{=} \\ &= A\left(\overset{s-3}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(e\left(a_1^{(s-2)m}\right), \varphi\left(b_1^m\right), a_1^{(s-2)m}\right), \overset{2}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{(b)}{=} A\left(\overset{s-3}{\overline{e\left(a_1^{(s-2)m}\right)}}, \varphi\left(\varphi\left(b_1^m\right)\right), \overset{2}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{1.3}{=} \\ &= A\left(\overset{s-3}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(\varphi^2\left(b_1^m\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right), \overset{2}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{1.1}{=} \\ &= A\left(\overset{s-4}{\overline{e\left(a_1^{(s-2)m}\right)}}, A\left(e\left(a_1^{(s-2)m}\right), \varphi^2\left(b_1^m\right), a_1^{(s-2)m}\right), \overset{3}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{(b)}{=} A\left(\overset{s-4}{\overline{e\left(a_1^{(s-2)m}\right)}}, \varphi^3\left(b_1^m\right), \overset{3}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) = \ldots = \\ &= A\left(\varphi^{s-1}\left(b_1^m\right), \overset{s-1}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{1.3}{=} A\left(A\left(\varphi^{s-1}\left(b_1^m\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right), \overset{s-1}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{1.1}{=} \\ &= A\left(\varphi^{s-1}\left(b_1^m\right), a_1^{(s-2)m}, A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right)\right) \overset{(c)}{=} A\left(\varphi^{s-1}\left(b_1^m\right), a_1^{(s-2)m}, c_1^m\right) \overset{(a)}{=} \varphi^{s-1}\left(b_1^m\right) \cdot c_1^m. \end{aligned}$$

(v) $\forall x_1^{sm} \in Q$ the following sequence of equalities holds:

$$
\begin{aligned}
& A\left(x_1^{sm}\right) \overset{1.6}{=} A\left(x_1^{(s-1)m}, A\left(a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right), x_{(s-1)m+1}^{sm}\right)\right) \overset{1.3}{=} \\
& = A\left(x_1^{(s-1)m}, A\left(a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right), A\left(x_{(s-1)m+1}^{sm}, a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right)\right)\right) \overset{1.1}{=} \\
& = A\left(x_1^{(s-1)m}, A\left(a_1^{(s-2)m}, A\left(e\left(a_1^{(s-2)m}\right), x_{(s-1)m+1}^{sm}, a_1^{(s-2)m}\right), e\left(a_1^{(s-2)m}\right)\right)\right) \overset{(b)}{=} \\
& = A\left(x_1^{(s-1)m}, A\left(a_1^{(s-2)m}, \varphi\left(x_{(s-1)m+1}^{sm}\right), e\left(a_1^{(s-2)m}\right)\right)\right) \overset{1.1}{=} \\
& = A\left(x_1^{(s-2)m}, A\left(x_{(s-2)m+1}^{(s-1)m}, a_1^{(s-2)m}, \varphi\left(x_{(s-1)m+1}^{sm}\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{(a)}{=} \\
& = A\left(x_1^{(s-2)m}, x_{(s-2)m+1}^{(s-1)m} \cdot \varphi\left(x_{(s-1)m+1}^{sm}\right), e\left(a_1^{(s-2)m}\right)\right) \overset{1.3}{=} \\
& = A\left(x_1^{(s-2)m}, A\left(x_{(s-2)m+1}^{(s-1)m} \cdot \varphi\left(x_{(s-1)m+1}^{sm}\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{1.6}{=} \\
& = A\left(x_1^{(s-2)m}, A\left(a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right), A\left(x_{(s-2)m+1}^{(s-1)m} \cdot \varphi\left(x_{(s-1)m+1}^{sm}\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{1.1}{=} \\
& = A\left(x_1^{(s-2)m}, A\left(a_1^{(s-2)m}, A\left(e\left(a_1^{(s-2)m}\right), x_{(s-2)m+1}^{(s-1)m} \cdot \varphi\left(x_{(s-1)m+1}^{sm}\right), a_1^{(s-2)m}\right), e\left(a_1^{(s-2)m}\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{(b)}{=} \\
& = A\left(x_1^{(s-2)m}, A\left(a_1^{(s-2)m}, \varphi\left(x_{(s-2)m+1}^{(s-1)m} \cdot \varphi\left(x_{(s-1)m+1}^{sm}\right)\right), e\left(a_1^{(s-2)m}\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{(ii)}{=} \\
& = A\left(x_1^{(s-2)m}, A\left(a_1^{(s-2)m}, \varphi\left(x_{(s-2)m+1}^{(s-1)m}\right) \cdot \varphi^2\left(x_{(s-1)m+1}^{sm}\right), e\left(a_1^{(s-2)m}\right)\right), e\left(a_1^{(s-2)m}\right)\right) \overset{1.1}{=} \\
& = A\left(x_1^{(s-3)m}, A\left(x_{(s-3)m+1}^{(s-2)m}, a_1^{(s-2)m}, \varphi\left(x_{(s-2)m+1}^{(s-1)m}\right) \cdot \varphi^2\left(x_{(s-1)m+1}^{sm}\right)\right), \overset{2}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{(a)}{=} \\
& = A\left(x_1^{(s-3)m}, x_{(s-3)m+1}^{(s-2)m} \cdot \varphi\left(x_{(s-2)m+1}^{(s-1)m}\right) \cdot \varphi^2\left(x_{(s-1)m+1}^{sm}\right), \overset{2}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) = \cdots = \\
& = A\left(x_1^{m} \cdot \varphi\left(x_{m+1}^{2m}\right) \cdot \varphi^2\left(x_{2m+1}^{3m}\right) \cdots \varphi^{s-1}\left(x_{(s-1)m+1}^{sm}\right), \overset{s-1}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{1.3}{=} \\
& = A\left(A\left(x_1^{m} \cdot \varphi\left(x_{m+1}^{2m}\right) \cdot \varphi^2\left(x_{2m+1}^{3m}\right) \cdots \varphi^{s-1}\left(x_{(s-1)m+1}^{sm}\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right), \overset{s-1}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{1.1}{=} \\
& = A\left(x_1^{m} \cdot \varphi\left(x_{m+1}^{2m}\right) \cdot \varphi^2\left(x_{2m+1}^{3m}\right) \cdots \varphi^{s-1}\left(x_{(s-1)m+1}^{sm}\right), a_1^{(s-2)m}, A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right)\right) \overset{(a),(c)}{=} \\
& = x_1^{m} \cdot \varphi\left(x_{m+1}^{2m}\right) \cdot \varphi^2\left(x_{2m+1}^{3m}\right) \cdots \varphi^{s-1}\left(x_{(s-1)m+1}^{sm}\right) \cdot c_1^m. \quad \square
\end{aligned}
$$

Theorem 2.3. *Let (Q, A) be an (sm, m)-group, $s \geq 3$, $e : Q^{(s-2)m} \to Q^m$ its $\{1, (s-1)m+1\}$-neutral operation. Also, let $(Q^m, \cdot)$ be a group and let for every sequence $x_1^{sm} \in Q$ holds:*

(a) $A\left(x_1^{sm}\right) = x_1^m \cdot \varphi\left(x_{m+1}^{2m}\right) \cdot \varphi^2\left(x_{2m+1}^{3m}\right) \cdots \varphi^{s-1}\left(x_{(s-1)m+1}^{sm}\right) \cdot c_1^m$, *where:*
(b) $\varphi \in Aut\left(Q^m, \cdot\right)$,
(c) $\varphi\left(c_1^m\right) = c_1^m$,
(d) $\varphi^{s-1}\left(b_1^m\right) \cdot c_1^m = c_1^m \cdot b_1^m$, $\forall b_1^m \in Q^m$.

Then, there is a sequence $a_1^{(s-2)m} \in Q$, such that $\forall x_1^m, y_1^m \in Q^m$ the following equalities hold:

(i) $x_1^m \cdot y_1^m = A\left(x_1^m, a_1^{(s-2)m}, y_1^m\right)$,
(ii) $\varphi\left(x_1^m\right) = A\left(e\left(a_1^{(s-2)m}\right), x_1^m, a_1^{(s-2)m}\right)$,

(iii) $c_1^m = A\left(\overline{\overline{e\left(a_1^{(s-2)m}\right)}}^{\,s}\right)$.

Proof. (i) Let $e_1^m \in Q^m$ be a neutral element of the group $(Q^m, \cdot)$ and $f : Q^m \to Q^m$ its inverse operation. Then, $\forall x_1^m, y_1^m \in Q^m$ the following sequence of equalities holds:

$$A\left(x_1^m, \overset{s-3}{\overline{e_1^m}}, f\left(c_1^m\right), y_1^m\right) \overset{(a)}{=} x_1^m \cdot \varphi\left(e_1^m\right) \cdots \varphi^{s-3}\left(e_1^m\right) \cdot \varphi^{s-2}\left(f\left(c_1^m\right)\right) \cdot \varphi^{s-1}\left(y_1^m\right) \cdot c_1^m \overset{(b)}{=}$$
$$= x_1^m \cdot \varphi^{s-2}\left(f\left(c_1^m\right)\right) \cdot \varphi^{s-1}\left(y_1^m\right) \cdot c_1^m \overset{(d)}{=} x_1^m \cdot \varphi^{s-2}\left(f\left(c_1^m\right)\right) \cdot c_1^m \cdot y_1^m = x_1^m \cdot e_1^m \cdot y_1^m = x_1^m \cdot y_1^m.$$

Hence, there is a sequence

$$a_1^{(s-2)m} = \left(\overset{s-3}{\overline{e_1^m}}, f\left(c_1^m\right)\right), \tag{1}$$

such that the following equality holds:

$$A\left(x_1^m, a_1^{(s-2)m}, y_1^m\right) = x_1^m \cdot y_1^m.$$

(ii) By (i) the following equality holds:

$$e_1^m \cdot e_1^m = A\left(e_1^m, a_1^{(s-2)m}, e_1^m\right).$$

By definition of the $\{1, (s-1)m+1\}$-neutral operation of the (sm, m)-group $(Q; A)$, the equality:

$$e_1^m = A\left(e\left(a_1^{(s-2)m}\right), a_1^{(s-2)m}, e_1^m\right)$$

holds.
Using the two above equalities and the assumption that (Q, A) is an (sm, m)-quasigroup, we conclude that the equality

$$e\left(a_1^{(s-2)m}\right) = e_1^m \tag{2}$$

holds. Then, by the following sequence of equalities, we conclude that there is a sequence $a_1^{(s-2)m} \in Q$ such that $\forall x_1^m \in Q^m$ the equality (ii) holds.

$$A\left(e\left(a_1^{(s-2)m}\right), x_1^m, a_1^{(s-2)m}\right) \overset{(1),(2)}{=} A\left(e_1^m, x_1^m, \overset{s-3}{\overline{e_1^m}}, f\left(c_1^m\right)\right) \overset{(a)}{=}$$
$$= e_1^m \cdot \varphi\left(x_1^m\right) \cdot \varphi^2\left(e_1^m\right) \cdots \varphi^{s-2}\left(e_1^m\right) \cdot \varphi^{s-1}\left(f\left(c_1^m\right)\right) \cdot c_1^m \overset{(b),(c)}{=} \varphi\left(x_1^m\right) \cdot f\left(c_1^m\right) \cdot c_1^m = \varphi\left(x_1^m\right).$$

(iii)
$$A\left(\overset{s}{\overline{e\left(a_1^{(s-2)m}\right)}}\right) \overset{(2)}{=} A\left(\overset{s}{\overline{e_1^m}}\right) \overset{(a)}{=} e_1^m \cdot \varphi\left(e_1^m\right) \cdots \varphi^{s-1}\left(e_1^m\right) \cdot c_1^m \overset{(b)}{=} c_1^m. \quad \square$$

For the algebra $\left(Q^m, \left\{\cdot, \varphi, c_1^m\right\}\right)$ which have been described in Theorems 2.2 and 2.3, we say that it is associated to the (sm, m)-group (Q, A).

Example 2.4. *Let (Q,A) be a $(6,2)$-group defined in Example 1.2 and $e : Q^2 \to Q^2$ be its neutral operation (see Example 1.4). For an arbitrary sequence $a_1^2 \in Q$ and $\forall x_1^2, y_1^2 \in Q^2$, we define the following operations:*

a) $x_1^2 * y_1^2 \overset{def}{=} A(x_1^2, a_1^2, y_1^2) = (x_1 \cdot \psi(a_1) \cdot y_1, x_2 \cdot \psi(a_2) \cdot y_2)$;

b) $\varphi(x_1^2) \overset{def}{=} A(e(a_1^2), x_1^2, a_1^2) = A(\psi(a_1), \psi(a_2), x_1^2, a_1^2) = (\psi(a_1) \cdot \psi(x_1) \cdot a_1, \psi(a_2) \cdot \psi(x_2) \cdot a_2)$;

c) $c_1^2 \overset{def}{=} A\left(\overset{3}{\overline{e\left(a_1^2\right)}}\right) = A(\psi(a_1), \psi(a_2), \psi(a_1), \psi(a_2), \psi(a_1), \psi(a_2)) = (\psi(a_1) \cdot a_1 \cdot \psi(a_1), \psi(a_2) \cdot a_2 \cdot \psi(a_2)) = (a_1, a_2)$.

Since the afore mentioned operators satisfy the assumptions of Theorem 2.2, the following statements hold: $(Q^2, *)$ is a group, $\varphi \in Aut(Q^2, *)$, $\varphi(c_1^2) = c_1^2$, $\varphi^2(b_1^2) * c_1^2 = c_1^2 * b_1^2$, $\forall b_1^2 \in Q^2$, $A(x_1^6) = x_1^2 * \varphi(x_3^4) * \varphi^2(x_5^6) * c_1^2$, $\forall x_1^6 \in Q$. Therefore, $(Q^2, \{*, \varphi, c_1^2\})$ is the algebra which is associated to the $(6,2)$-group (Q,A).

In the following proposition we prove some interesting equalities which relate a $\{1, (s-1)m+1\}$-neutral operation of an (sm, m)-group (Q,A) and an inverse operation of the binary group $(Q^m, \cdot)$, which also relate an inverse operation of an (sm, m)-group (Q,A) and an inverse operation of the binary group $(Q^m, \cdot)$.

Proposition 2.5. *Let (Q,A) be an (sm, m)-group, $s \geq 3$, $e : Q^{(s-2)m} \to Q^m$ its $\{1, (s-1)m+1\}$-neutral operation and $^{-1} : Q^{(s-1)m} \to Q^m$ its inverse operation. Also, let $\left(Q^m, \left\{\cdot, \varphi, c_1^m\right\}\right)$ be an algebra associated to the (sm, m)-group (Q,A) and $f : Q^m \to Q^m$ inverse operation of the group $(Q^m, \cdot)$. Then, $\forall d_1^{(s-2)m} \in Q$ the following equality holds:*

$$e\left(d_1^{(s-2)m}\right) = f\left(\varphi\left(d_1^m\right) \cdot \varphi^2\left(d_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(d_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m\right). \tag{3}$$

Proof. Firstly, we will prove that there is a sequence $a_1^{(s-2)m} \in Q$ such that $\forall b_1^m \in Q^m$ the following equality holds:

$$f\left(b_1^m\right) = \left(a_1^{(s-2)m}, b_1^m\right)^{-1}. \tag{4}$$

$$b_1^m \cdot \left(a_1^{(s-2)m}, b_1^m\right)^{-1} \overset{2.3}{=} A\left(b_1^m, a_1^{(s-2)m}, \left(a_1^{(s-2)m}, b_1^m\right)^{-1}\right) \overset{1.5}{=} e\left(a_1^{(s-2)m}\right).$$

By Theorem 2.3 there is a sequence $a_1^{(s-2)m} = \left(\overset{s-3}{\overline{e_1^m}}, f\left(c_1^m\right)\right)$ such that $e\left(a_1^{(s-2)m}\right) = e_1^m$. Thus,

$$b_1^m \cdot \left(a_1^{(s-2)m}, b_1^m\right)^{-1} = e_1^m = b_1^m \cdot f\left(b_1^m\right) \Rightarrow \left(a_1^{(s-2)m}, b_1^m\right)^{-1} = f\left(b_1^m\right).$$

From the following sequence of equalities, we can conclude that $\forall d_1^{(s-2)m} \in Q$ the statement of the proposition holds:

$$x_1^m \cdot f\left(e\left(d_1^{(s-2)m}\right)\right) \overset{2.3}{=} A\left(x_1^m, a_1^{(s-2)m}, f\left(e\left(d_1^{(s-2)m}\right)\right)\right) \overset{(4)}{=} A\left(x_1^m, a_1^{(s-2)m}, \left(a_1^{(s-2)m}, e\left(d_1^{(s-2)m}\right)\right)^{-1}\right) \overset{1.3}{=}$$
$$= A\left(A\left(x_1^m, a_1^{(s-2)m}, \left(a_1^{(s-2)m}, e\left(d_1^{(s-2)m}\right)\right)^{-1}\right), a_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right) \overset{1.7}{=} A\left(x_1^m, d_1^{(s-2)m}, e\left(a_1^{(s-2)m}\right)\right) \overset{(2)}{=}$$
$$= A\left(x_1^m, d_1^{(s-2)m}, e_1^m\right) \overset{2.2}{=} x_1^m \cdot \varphi\left(d_1^m\right) \cdot \varphi^2\left(d_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(d_{(s-3)m+1}^{(s-2)m}\right) \cdot \varphi^{s-1}\left(e_1^m\right) \cdot c_1^m \overset{2.2}{=}$$
$$= x_1^m \cdot \varphi\left(d_1^m\right) \cdot \varphi^2\left(d_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(d_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m. \quad \square$$

3. A Central Operation in Terms of a Hosszú-Gluskin Algebra for an (sm, m) − *Group*

Further aim of research considering (n,m)-structures was to generalize the notion of a central element in a binary group, i.e. to define mapping whose properties for $(n,m) = (2,1)$ would correspond to properties of the central element in a binary group. In n-group this notion was described by Ušan in 2001 [15].

Definition 3.1. *[7] Let (Q, A) be an (n, m)-group, $n \geq 2m$ and let α be a map of the set Q^{n-2m} into the set Q^m. We say that α is a central operation of an (n, m)-group (Q, A) iff $\forall a_1^{n-2m} \in Q$, $\forall b_1^{n-2m} \in Q$ and $\forall x_1^m \in Q^m$ the following equality holds:*

$$A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = A\left(x_1^m, \alpha\left(b_1^{n-2m}\right), b_1^{n-2m}\right). \tag{5}$$

By Proposition 1.6 we see that a $\{1, n-m+1\}$-neutral operation is an example of the central operation of the (n, m)-group.

By means of a series of propositions, in [7] it was proved that a mapping $\alpha : Q^{n-2m} \to Q^m$ is a central operation of the (n, m)-group (Q, A) iff the equality

$$A\left(x_1^m, a_1^{n-2m}, \alpha\left(a_1^{n-2m}\right)\right) = A\left(b_1^{n-2m}, \alpha\left(b_1^{n-2m}\right), x_1^m\right)$$

holds.

Proposition 3.2. *Let (Q, A) be an (n, m)-group, $n \geq 2m$ and $\alpha, \beta : Q^{n-2m} \to Q^m$ its central operations. Then, $\forall a_1^{n-2m} \in Q$ the following equality holds:*

$$A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, \beta\left(a_1^{n-2m}\right)\right) = A\left(\beta(a_1^{n-2m}), a_1^{n-2m}, \alpha(a_1^{n-2m})\right).$$

Proof. For the central operation α of the (n, m)-group (Q, A) the equality

$$A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = A\left(x_1^m, b_1^{n-2m}, \alpha\left(b_1^{n-2m}\right)\right)$$

holds $\forall a_1^{n-2m}, b_1^{n-2m}, x_1^m \in Q$ (see [7]). The proof of the proposition follows directly from the previous equality if we put x_1^m instead of $\beta(a_1^{n-2m})$ and b_1^{n-2m} instead of a_1^{n-2m}. □

Theorem 3.3. *Let (Q, A) be an (n, m)-group, $n \geq 2m$ and α its central operation. Then, there is a bijection $\sigma_\alpha : Q^m \to Q^m$ such that $\forall x_1^m \in Q^m$ and $\forall a_1^{n-2m}, b_1^{n-2m} \in Q$ the following equalities hold:*

(i) $A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = \sigma_\alpha\left(x_1^m\right)$,
(ii) $A\left(x_1^m, \alpha\left(b_1^{n-2m}\right), b_1^{n-2m}\right) = \sigma_\alpha\left(x_1^m\right)$.

Proof. Let $k_1^{n-2m} \in Q$ be an arbitrary sequence. Then, we define:

$$A\left(x_1^m, \alpha\left(k_1^{n-2m}\right), k_1^{n-2m}\right) \stackrel{def}{=} \sigma_\alpha\left(x_1^m\right).$$

Firstly, we will prove that such a defined mapping σ_α is a bijection. Since (Q, A) is an (n, m)-quasigroup, by Definition 1.1 (b), for $i = 1$ holds: $\forall a_1^n \in Q$, $\exists! x_1^m \in Q^m$ such that holds $A\left(x_1^m, a_1^{n-m}\right) = a_{n-m+1}^n$.
If we replace the sequence a_1^n in the above statement with $A\left(\alpha\left(k_1^{n-2m}\right), k_1^{n-2m}, y_1^m\right)$, then the following statement holds: $\forall y_1^m \in Q$, $\exists! x_1^m \in Q^m$ such that holds:

$$y_1^m = A\left(x_1^m, \alpha\left(k_1^{n-2m}\right), k_1^{n-2m}\right) = \sigma_\alpha\left(x_1^m\right).$$

Then, equalities from the theorem hold by Definition 3.1. □

Definition 3.4. *Let (Q, A) be an (n, m)-group, $n \geq 2m$, $\alpha : Q^{n-2m} \to Q^m$ its central operation and let $\sigma_\alpha : Q^m \to Q^m$ be a bijection. We say that the bijection σ_α is associated to the central operation α iff $\forall x_1^m \in Q^m$ and $\forall a_1^{n-2m} \in Q$ holds:*

$$A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right) = \sigma_\alpha\left(x_1^m\right). \tag{6}$$

Example 3.5. *Let $(Q,\cdot)$, $Q=\{1,2,3,4\}$ be the Klein group, let $\psi=\begin{pmatrix}1\,2\,3\,4\\1\,2\,4\,3\end{pmatrix}$ be the permutation of the set Q. In Example 1.2 we proved that (Q,A) is a $(6,2)$-group, when $A:Q^6\to Q^2$ is the mapping defined with $A(x_1^6)=(x_1\cdot\psi(x_3)\cdot x_5, x_2\cdot\psi(x_4)\cdot x_6)$. Let $\alpha:Q^2\to Q^2$ be a mapping such that $\forall x_1^2\in Q^2$ the following equality holds:*

$$\alpha(x_1^2)\overset{def}{=}(2\cdot\psi(x_1),2\cdot\psi(x_2)).$$

We prove that this mapping is a central operation of the $(6,2)$-group (Q,A). By Definition 3.1 we need to prove that $\forall a_1^2\in Q$, $\forall b_1^2\in Q$ and $\forall x_1^2\in Q^2$ the equality (5) holds for $m=2$, $n=6$.

$$\begin{aligned}&A(x_1^2,a_1^2,\alpha(a_1^2))=A(b_1^2,\alpha(b_1^2),x_1^2)\Leftrightarrow\\&\Leftrightarrow A(x_1,x_2,a_1,a_2,2\cdot\psi(a_1),2\cdot\psi(a_2))=A(b_1,b_2,2\cdot\psi(b_1),2\cdot\psi(b_2),x_1,x_2)\Leftrightarrow\\&\Leftrightarrow(x_1\cdot\psi(a_1)\cdot2\cdot\psi(a_1),x_2\cdot\psi(a_2)\cdot2\cdot\psi(a_2))=(b_1\cdot\psi(2)\cdot b_1\cdot x_1,b_2\cdot\psi(2)\cdot b_2\cdot x_2)\Leftrightarrow\\&\Leftrightarrow(2\cdot x_1,2\cdot x_2)=(2\cdot x_1,2\cdot x_2)\end{aligned}$$

With $\sigma_\alpha(x_1^2)\overset{def}{=}(2\cdot x_1,2\cdot x_2)$ a bijection $\sigma_\alpha:Q^2\to Q^2$ associated to the central operation α is defined. Let us prove equation (6):

$$A(\alpha(a_1^2),a_1^2,x_1^2)=A(2\cdot\psi(a_1),2\cdot\psi(a_2),a_1,a_2,x_1,x_2)=(2\cdot\psi(a_1)\cdot\psi(a_1)\cdot x_1,2\cdot\psi(a_2)\cdot\psi(a_2)\cdot x_2)=(2\cdot x_1,2\cdot x_2)=\sigma_\alpha(x_1^2).$$

Theorem 3.6. *Let (Q,A) be an (n,m)-group, $n\geq 2m$, $\alpha:Q^{n-2m}\to Q^m$ its central operation and let the bijection $\sigma_\alpha:Q^m\to Q^m$ be associated to the central operation α. Then, $\forall x_1^n\in Q$ the following equalities hold:*

$$(i)\ \sigma_\alpha\left(A\left(x_1^n\right)\right)=A\left(\sigma_\alpha\left(x_1^m\right),x_{m+1}^n\right),$$
$$(ii)\ \sigma_\alpha\left(A\left(x_1^n\right)\right)=A\left(x_1^m,\sigma_\alpha\left(x_{m+1}^{2m}\right),x_{2m+1}^n\right),$$
$$(iii)\ \sigma_\alpha\left(A\left(x_1^n\right)\right)=A\left(x_1^{n-m},\sigma_\alpha\left(x_{n-m+1}^n\right)\right).$$

Proof. (i)

$$\sigma_\alpha\left(A\left(x_1^n\right)\right)\overset{(6)}{=}A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},A\left(x_1^n\right)\right)\overset{1.1}{=}A\left(A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},x_1^m\right),x_{m+1}^n\right)\overset{(6)}{=}A\left(\sigma_\alpha\left(x_1^m\right),x_{m+1}^n\right).$$

(ii)

$$\begin{aligned}&\sigma_\alpha\left(A\left(x_1^n\right)\right)\overset{(6)}{=}A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},A\left(x_1^n\right)\right)\overset{1.1}{=}A\left(A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},x_1^m\right),x_{m+1}^n\right)\overset{(5)}{=}\\&=A\left(A\left(x_1^m,\alpha\left(a_1^{n-2m}\right),a_1^{n-2m}\right),x_{m+1}^n\right)\overset{1.1}{=}A\left(x_1^m,A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},x_{m+1}^{2m}\right),x_{2m+1}^n\right)\overset{(6)}{=}A\left(x_1^m,\sigma_\alpha\left(x_{m+1}^{2m}\right),x_{2m+1}^n\right).\end{aligned}$$

(iii)

$$\begin{aligned}&\sigma_\alpha\left(A\left(x_1^n\right)\right)\overset{(6)}{=}A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},A\left(x_1^n\right)\right)\overset{(5)}{=}A\left(A\left(x_1^n\right),\alpha\left(a_1^{n-2m}\right),a_1^{n-2m}\right)\overset{1.1}{=}\\&=A\left(x_1^{n-m},A\left(x_{n-m+1}^n,\alpha\left(a_1^{n-2m}\right),a_1^{n-2m}\right)\right)\overset{(5)}{=}A\left(x_1^{n-m},A\left(\alpha\left(a_1^{n-2m}\right),a_1^{n-2m},x_{n-m+1}^n\right)\right)\overset{(6)}{=}A\left(x_1^{n-m},\sigma_\alpha\left(x_{n-m+1}^n\right)\right).\end{aligned}$$ □

If we have two central operations of an (n,m)-group and if a bijection is associated to each of them as defined in 3.4, then the bijections commute.

Theorem 3.7. *Let (Q,A) be an (n,m)-group, $n\geq 2m$ and $\alpha,\beta:Q^{n-2m}\to Q^m$ its central operations. Also, let the bijection σ_α be associated to the central operation α and the bijection σ_β be associated to the central operation β. Then $\forall x_1^m\in Q^m$ the following equality holds:*

$$\sigma_\alpha\left(\sigma_\beta\left(x_1^m\right)\right)=\sigma_\beta\left(\sigma_\alpha\left(x_1^m\right)\right).$$

Proof. $\sigma_\alpha\left(\sigma_\beta\left(x_1^m\right)\right) \overset{(6)}{=} A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, \sigma_\beta\left(x_1^m\right)\right) \overset{(6)}{=} A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, A\left(\beta\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right)\right) \overset{1.1}{=}$
$= A\left(A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, \beta\left(a_1^{n-2m}\right)\right), a_1^{n-2m}, x_1^m\right) \overset{3.2}{=} A\left(A\left(\beta\left(a_1^{n-2m}\right), a_1^{n-2m}, \alpha\left(a_1^{n-2m}\right)\right), a_1^{n-2m}, x_1^m\right) \overset{1.1}{=}$
$= A\left(\beta\left(a_1^{n-2m}\right), a_1^{n-2m}, A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right)\right) \overset{(6)}{=} \sigma_\beta\left(\sigma_\alpha\left(x_1^m\right)\right).$ □

Under particular condition, a bijection associated to the central operation α is an involution which has been proved in the following theorem.

Theorem 3.8. *Let (Q, A) be an (n, m)-group, $n \geq 2m$, $\alpha : Q^{n-2m} \to Q^m$ its central operation, let σ_α be the bijection associated to the central operation α and let $^{-1} : Q^{n-m} \to Q^m$ be an inverse operation of the (n, m)-group (Q, A). If $\forall a_1^{n-2m} \in Q$ the equality*

$$\left(a_1^{n-2m}, \alpha\left(a_1^{n-2m}\right)\right)^{-1} = \alpha\left(a_1^{n-2m}\right) \tag{7}$$

holds, then $\forall x_1^m \in Q^m$ the following equality holds:

$$\sigma_\alpha\left(\sigma_\alpha\left(x_1^m\right)\right) = x_1^m.$$

Proof. $\forall a_1^{n-2m} \in Q$ the following sequence of equalities holds:
$\sigma_\alpha\left(\sigma_\alpha\left(x_1^m\right)\right) \overset{(6)}{=} A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, \sigma_\alpha\left(x_1^m\right)\right) \overset{(6)}{=} A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right)\right) \overset{(7)}{=}$
$= A\left(\left(a_1^{n-2m}, \alpha\left(a_1^{n-2m}\right)\right)^{-1}, a_1^{n-2m}, A\left(\alpha\left(a_1^{n-2m}\right), a_1^{n-2m}, x_1^m\right)\right) \overset{1.5}{=} x_1^m.$ □

Theorem 3.9. *Let (Q, A) be an (sm, m)-group, $s \geq 3$ and $\left(Q^m, \{\cdot, \varphi, c_1^m\}\right)$ an algebra associated to the (sm, m)-group (Q, A). Also, let f be an inverse operation in a group $(Q^m, \cdot)$, $\alpha : Q^{(s-2)m} \to Q^m$ central operation of the (sm, m)-group (Q, A) and σ_α a bijection associated to the central operation α. Then, $\exists! y_1^m \in Q^m$ such that for every sequence $a_1^{(s-2)m} \in Q$ and $\forall x_1^m \in Q^m$ the following equalities hold:*

(i) $\alpha\left(a_1^{(s-2)m}\right) = y_1^m \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right)$,
(ii) $\sigma_\alpha\left(x_1^m\right) = \left(y_1^m \cdot c_1^m\right) \cdot x_1^m$,
(iii) $\varphi\left(y_1^m\right) = y_1^m$,
(iv) $\left(y_1^m \cdot c_1^m\right) \cdot x_1^m = x_1^m \cdot \left(y_1^m \cdot c_1^m\right)$.

Proof. (i) Let us prove that by assumptions of the theorem $\exists! y_1^m \in Q^m$ such that for every sequence $a_1^{(s-2)m} \in Q$ the following equality holds:

$$\alpha\left(a_1^{(s-2)m}\right) \cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) = y_1^m.$$

Let $k_1^{(s-2)m} \in Q$ be an arbitrary fixed sequence. Then by the definition of the central operation of an (sm, m)-group, $\forall x_1^m \in Q^m$ i $\forall a_1^{(s-2)m} \in Q$ holds:

$$A\left(\alpha\left(a_1^{(s-2)m}\right), a_1^{(s-2)m}, x_1^m\right) = A\left(x_1^m, \alpha\left(k_1^{(s-2)m}\right), k_1^{(s-2)m}\right) = A\left(\alpha\left(k_1^{(s-2)m}\right), k_1^{(s-2)m}, x_1^m\right).$$

Moreover, by Theorem 2.2 $\forall x_1^m \in Q^m$, $\forall a_1^{(s-2)m} \in Q$ the following sequence of equivalences holds:

$A\left(\alpha\left(a_1^{(s-2)m}\right), a_1^{(s-2)m}, x_1^m\right) = A\left(\alpha\left(k_1^{(s-2)m}\right), k_1^{(s-2)m}, x_1^m\right) \Leftrightarrow$
$\alpha\left(a_1^{(s-2)m}\right) \cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot \varphi^{s-1}\left(x_1^m\right) \cdot c_1^m =$
$= \alpha\left(k_1^{(s-2)m}\right) \cdot \varphi\left(k_1^m\right) \cdot \varphi^2\left(k_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(k_{(s-3)m+1}^{(s-2)m}\right) \cdot \varphi^{s-1}\left(x_1^m\right) \cdot c_1^m \Leftrightarrow$

$$\alpha\left(a_1^{(s-2)m}\right)\cdot\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)=\alpha\left(k_1^{(s-2)m}\right)\cdot\varphi\left(k_1^m\right)\cdot\varphi^2\left(k_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(k_{(s-3)m+1}^{(s-2)m}\right).$$

Because $k_1^{(s-2)m}\in Q$ is a fixed sequence, we can denote:

$$\alpha\left(k_1^{(s-2)m}\right)\cdot\varphi\left(k_1^m\right)\cdot\varphi^2\left(k_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(k_{(s-3)m+1}^{(s-2)m}\right)=y_1^m,$$

which, due to the last equality from the above sequence of equalities yields the equality:

$$\alpha\left(a_1^{(s-2)m}\right)\cdot\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)=y_1^m.$$

Because f is an inverse operation in the group $(Q^m,\cdot)$, the last equality is equivalent to:

$$\alpha\left(a_1^{(s-2)m}\right)=y_1^m\cdot f\left(\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right).$$

(*ii*) $\exists! y_1^m\in Q^m$ such that $\forall x_1^m\in Q^m$ i $\forall a_1^{(s-2)m}\in Q$ the following sequence of equalities holds:

$$\sigma_\alpha\left(x_1^m\right)\overset{(6)}{=}A\left(\alpha\left(a_1^{(s-2)m}\right),a_1^{(s-2)m},x_1^m\right)\overset{2.2}{=}\alpha\left(a_1^{(s-2)m}\right)\cdot\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\cdot\varphi^{s-1}\left(x_1^m\right)\cdot c_1^m\overset{2.2}{=}$$
$$=\alpha\left(a_1^{(s-2)m}\right)\cdot\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\cdot c_1^m\cdot x_1^m\overset{(i)}{=}\left(y_1^m\cdot c_1^m\right)\cdot x_1^m.$$

(*iii*) $\forall a_1^{(s-2)m},k_1^{(s-2)m}\in Q$ and $\forall x_1^m\in Q^m$, by definition of a central operation and Theorem 2.2, the following sequence of equivalences holds:

$$A\left(\alpha\left(a_1^{(s-2)m}\right),a_1^{(s-2)m},x_1^m\right)=A\left(x_1^m,\alpha\left(k_1^{(s-2)m}\right),k_1^{(s-2)m}\right)\Leftrightarrow$$
$$\alpha\left(a_1^{(s-2)m}\right)\cdot\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\cdot c_1^m\cdot x_1^m=x_1^m\cdot\varphi\left(\alpha\left(k_1^{(s-2)m}\right)\right)\cdot\varphi^2\left(k_1^m\right)\cdots\varphi^{s-1}\left(k_{(s-3)m+1}^{(s-2)m}\right)\cdot c_1^m\Leftrightarrow$$
$$\alpha\left(a_1^{(s-2)m}\right)\cdot\varphi\left(a_1^m\right)\cdot\varphi^2\left(a_{m+1}^{2m}\right)\cdots\varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\cdot c_1^m\cdot x_1^m=x_1^m\cdot\varphi\left(\alpha\left(k_1^{(s-2)m}\right)\cdot\varphi\left(k_1^m\right)\cdots\varphi^{s-2}\left(k_{(s-3)m+1}^{(s-2)m}\right)\right)\cdot c_1^m.$$

By (*i*), the last equality is equivalent to:

$$y_1^m\cdot c_1^m\cdot x_1^m=x_1^m\cdot\varphi\left(y_1^m\right)\cdot c_1^m.$$

Since this equality holds $\forall x_1^m\in Q^m$, thus it holds for $x_1^m=e_1^m$ where e_1^m is a neutral element of a group $(Q^m,\cdot)$. Accordingly, it follows:

$$y_1^m\cdot c_1^m=\varphi\left(y_1^m\right)\cdot c_1^m,$$

that is

$$y_1^m=\varphi\left(y_1^m\right).$$

(*iv*) In the proof (*iii*), we have proved that $\exists! y_1^m\in Q^m$ such that $\forall x_1^m\in Q^m$ the following equalities hold:

$$y_1^m\cdot c_1^m\cdot x_1^m=x_1^m\cdot\varphi\left(y_1^m\right)\cdot c_1^m$$

and

$$y_1^m=\varphi\left(y_1^m\right).$$

From the above equalities it follows:

$$\left(y_1^m\cdot c_1^m\right)\cdot x_1^m=x_1^m\cdot\left(y_1^m\cdot c_1^m\right).$$

□

Theorem 3.10. *Let (Q, A) be an (sm, m)-group, $s \geq 3$ and $\left(Q^m, \{\cdot, \varphi, c_1^m\}\right)$ an algebra which is associated to the (sm, m)-group (Q, A). Also, let f be an inverse operation in the group $(Q^m, \cdot)$ and $e_1^m \in Q^m$ its neutral element. Let α be a central operation of the (sm, m)-group (Q, A), σ_α a bijection associated to the central operation α and let ${}^{-1} : Q^{(s-1)m} \to Q^m$ be an inverse operation of the (sm, m)-group (Q, A). If for every sequence $a_1^{(s-2)m} \in Q$ the equality*

$$\left(a_1^{(s-2)m}, \alpha\left(a_1^{(s-2)m}\right)\right)^{-1} = \alpha\left(a_1^{(s-2)m}\right)$$

holds, then $\exists! y_1^m \in Q^m$ such that the following equality holds:

$$\left(y_1^m \cdot c_1^m\right) \cdot \left(y_1^m \cdot c_1^m\right) = e_1^m.$$

Proof. $\forall x_1^m, a_1^{(s-2)m} \in Q$, $\exists! y_1^m \in Q^m$ such that the following sequence of equalities holds:
$x_1^m \overset{3.8}{=} \sigma_\alpha\left(\sigma_\alpha\left(x_1^m\right)\right) \overset{3.9}{=} \sigma_\alpha\left(\left(y_1^m \cdot c_1^m\right) \cdot x_1^m\right) \overset{3.9}{=} \left(y_1^m \cdot c_1^m\right) \cdot \left(y_1^m \cdot c_1^m\right) \cdot x_1^m.$

Since $(Q^m, \cdot)$ is a group, from the above sequence of equalities it follows:
$\left(y_1^m \cdot c_1^m\right) \cdot \left(y_1^m \cdot c_1^m\right) = e_1^m.$ □

Theorem 3.11. *Let (Q, A) be an (sm, m)-group, $s \geq 3$ and $\left(Q^m, \{\cdot, \varphi, c_1^m\}\right)$ the algebra associated to the (sm, m)-group (Q, A). Also, let f be an inverse operation of the group $(Q^m, \cdot)$ and $e_1^m \in Q^m$ its neutral element. Let $y_1^m \in Q^m$ be a fixed sequence such that $\forall x_1^m \in Q^m$ the following equalities hold:*

(a) $\left(y_1^m \cdot c_1^m\right) \cdot x_1^m = x_1^m \cdot \left(y_1^m \cdot c_1^m\right)$,
(b) $\varphi\left(y_1^m\right) = y_1^m$,
(c) $\left(y_1^m \cdot c_1^m\right) \cdot \left(y_1^m \cdot c_1^m\right) = e_1^m$.
We define the mapping $\alpha : Q^{(s-2)m} \to Q^m$ such that $\forall a_1^{(s-2)m} \in Q$ holds:
(d) $\alpha\left(a_1^{(s-2)m}\right) \overset{def}{=} y_1^m \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{(2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right)$.

Then, the following statements hold:

(i) α is a central operation of the (sm, m)-group (Q, A);
(ii) for every sequence $a_1^{(s-2)m} \in Q$ the equality

$$\left(a_1^{(s-2)m}, \alpha\left(a_1^{(s-2)m}\right)\right)^{-1} = \alpha\left(a_1^{(s-2)m}\right)$$

holds, where ${}^{-1} : Q^{(s-1)m} \to Q^m$ is an inverse operation of the (sm, m)-group (Q, A).

Proof. *(i)* $\forall a_1^{(s-2)m} \in Q$ i $\forall x_1^m \in Q^m$ the following sequence of equalities holds:

$$A\left(\alpha\left(a_1^{(s-2)m}\right), a_1^{(s-2)m}, x_1^m\right) \overset{2.2}{=} \alpha\left(a_1^{(s-2)m}\right) \cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \cdot x_1^m \overset{(d)}{=}$$
$$= y_1^m \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right) \cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \cdot x_1^m =$$
$$= \left(y_1^m \cdot c_1^m\right) \cdot x_1^m.$$

Further on, $\forall b_1^{(s-2)m} \in Q$ i $\forall x_1^m \in Q^m$ the following sequence of equalities holds:

$$A\left(x_1^m, \alpha\left(b_1^{(s-2)m}\right), b_1^{(s-2)m}\right) \overset{2.2}{=} x_1^m \cdot \varphi\left(\alpha\left(b_1^{(s-2)m}\right)\right) \cdot \varphi^2\left(b_1^m\right) \cdots \varphi^{s-1}\left(b_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \overset{2.2}{=}$$
$$= x_1^m \cdot \varphi\left(\alpha\left(b_1^{(s-2)m}\right) \cdot \varphi\left(b_1^m\right) \cdot \varphi^2\left(b_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(b_{(s-3)m+1}^{(s-2)m}\right)\right) \cdot c_1^m \overset{(d)}{=}$$

$$= x_1^m \cdot \varphi\left(y_1^m \cdot f\left(\varphi\left(b_1^m\right) \cdot \varphi^2\left(b_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(b_{(s-3)m+1}^{(s-2)m}\right)\right) \cdot \varphi\left(b_1^m\right) \cdot \varphi^2\left(b_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(b_{(s-3)m+1}^{(s-2)m}\right)\right) \cdot c_1^m =$$
$$= x_1^m \cdot \varphi\left(y_1^m\right) \cdot c_1^m \overset{(b)}{=} x_1^m \cdot \left(y_1^m \cdot c_1^m\right) \overset{(a)}{=} \left(y_1^m \cdot c_1^m\right) \cdot x_1^m.$$

From the previous two sequences of equalities, we conclude that the statement (i) of the theorem holds.

(ii) By Proposition 1.5, $\forall a_1^{(s-2)m} \in Q$ and $\forall x_1^m \in Q^m$ the equality

$$A\left(\left(a_1^{(s-2)m}, x_1^m\right)^{-1}, a_1^{(s-2)m}, x_1^m\right) = e\left(a_1^{(s-2)m}\right)$$

holds, where $e : Q^{(s-2)m} \to Q^m$ is a $\{1,(s-1)m+1\}$-neutral operation of the (sm, m)-group (Q, A). For $x_1^m = \alpha\left(a_1^{(s-2)m}\right)$, by Theorem 2.2 and Proposition 2.5, the above equality is:

$$\left(a_1^{(s-2)m}, \alpha\left(a_1^{(s-2)m}\right)\right)^{-1} \cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \cdot \alpha\left(a_1^{(s-2)m}\right) =$$
$$= f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m\right),$$

which implies the following sequence of equalities:

$$\left(a_1^{(s-2)m}, \alpha\left(a_1^{(s-2)m}\right)\right)^{-1} =$$
$$= f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m\right) \cdot f\left(\alpha\left(a_1^{(s-2)m}\right)\right) \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m\right) =$$
$$= f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \cdot \alpha\left(a_1^{(s-2)m}\right) \cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m\right) \overset{(d)}{=}$$
$$= f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \cdot y_1^m \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right) \cdot\right.$$
$$\left.\cdot \varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m\right) = f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right) \cdot c_1^m \cdot y_1^m \cdot c_1^m\right) =$$
$$= f\left(c_1^m\right) \cdot f\left(y_1^m\right) \cdot f\left(c_1^m\right) \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right) =$$
$$= f\left(c_1^m\right) \cdot f\left(y_1^m\right) \cdot f\left(c_1^m\right) \cdot f\left(y_1^m\right) \cdot y_1^m \cdot f\left(\varphi\left(a_1^m\right) \cdot \varphi^2\left(a_{m+1}^{2m}\right) \cdots \varphi^{s-2}\left(a_{(s-3)m+1}^{(s-2)m}\right)\right) \overset{(c),(d)}{=}$$
$$= e_1^m \cdot \alpha\left(a_1^{(s-2)m}\right) = \alpha\left(a_1^{(s-2)m}\right). \quad \square$$

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